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Poisson Distribution

Poisson Distribution is a limiting form Binomial distribution as $n \to \infty$ and $p \to 0$. is such a way that np = a finite constant λ.

A random variable x taking non-negative values said to follow poisson distribution if its probability mass function is given by

$$P[X=x] = \frac{e^{-\lambda}\lambda^{x}}{x!}, \quad x = 0, 1, 2 \dots \infty$$

This distribution has only one parameter $\lambda > 0$.

Derive MGF, mean & variance of Poisson Distribution.

The probability Mass function of ~~probability~~ poisson distribution is $P[X=x] = \frac{e^{-\lambda}\lambda^{x}}{x!}$

MGF

$$M_X(t) = E[e^{tx}] = \sum_{x=0}^{\infty} e^{tx} P(x)$$

$$= \sum_{x=0}^{\infty} e^{tx} \frac{e^{-\lambda}\lambda^{x}}{x!}$$

$$= e^{-\lambda} \sum_{x=0}^{\infty} \frac{(\lambda e^{t})^{x}}{x!}$$

$$= e^{-\lambda}\left[1 + \frac{\lambda e^t}{1!} + \frac{(\lambda e^t)^2}{2!} + \cdots\right].$$

$$= e^{-\lambda}\, e^{\lambda e^t} \qquad \left(\because e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots\right)$$

$$\boxed{M_x(t) = e^{\lambda(e^t - 1)}}$$

$$\text{Mean} = \left[M_x'(t)\right]_{t=0} = E[x(t)]$$

$$= \left[\frac{d}{dt}\, e^{-\lambda} e^{\lambda e^t}\right]_{t=0}.$$

$$\frac{d}{dt} e^{\lambda e^t}, \quad \text{put } u = \lambda e^t$$

$$du = \lambda e^t dt$$

$$\frac{du}{dt} = \lambda e^t.$$

$$= e^{-\lambda}\left[\frac{d}{dt}\, e^{\lambda e^t}\right]_{t=0}.$$

$$\therefore \frac{d}{dt} e^{\lambda e^t} = \frac{d}{dt}(e^u)$$

$$= \left[e^{-\lambda}\, e^{\lambda e^t} . \lambda e^t\right]_{t=0}.$$

$$= e^u \frac{du}{dt}$$

$$= e^{\lambda e^t} \lambda e^t$$

$$= e^{-\lambda} e^{\lambda} \lambda$$

$$\boxed{\text{Mean} = \lambda.}$$

$$E[x^2(t)] = \left[M_x''(t)\right]_{t=0}.$$

$$= e^{-\lambda}\left\{\frac{d}{dt}\left[\lambda e^t e^{\lambda e^t}\right]\right\}_{t=0}$$

$$= \lambda e^{-\lambda}\left[\frac{d}{dt}\left(e^t e^{\lambda e^t}\right)\right]_{t=0}.$$

$$= \lambda e^{-\lambda}\left\{e^t e^{\lambda e^t} . \lambda e^t + e^{\lambda e^t} e^t\right\}_{t=0}.$$

$= \lambda e^{-\lambda} \{ \lambda e^{\lambda} + e^{\lambda} \}$

$= \lambda^2 e^{-\lambda} e^{\lambda} + \lambda e^{-\lambda} e^{\lambda} = \lambda^2 + \lambda.$

$E[x^2] = \lambda^2 + \lambda.$

$\therefore$ Variance $= E[x^2] - (E[x])^2$

$= \lambda^2 + \lambda - \lambda^2.$

$\boxed{\text{Variance} = \lambda}$

Remark

For poisson distribution Mean = Variance

Additive Property

$M_{x+y}(t) = M_x(t) M_y(t)$

$= e^{\lambda_1 (e^t - 1)} e^{\lambda_2 (e^t - 1)}.$

$= e^{\lambda_1 + \lambda_2 (e^t - 1)}.$

Problems

1. If x is a poisson variant such that $E[x^2] = 6$. find $E(x)$

Solution

Gn). $E(x^2) = 6$

$\Rightarrow \lambda^2 + \lambda = 6 \Rightarrow \lambda^2 + \lambda - 6 = 0$

$\Rightarrow (\lambda + 3)(\lambda - 2) = 0.$

$\Rightarrow \lambda = -3, 2.$

$\Rightarrow E(x) = \lambda = 2.$ ($\lambda = -3$ is impossible $\because \lambda > 0$)

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